INTRODUCTION TO TRIGONOMETRY 8

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry.

– J.F. Herbart (1890)

8.1 Introduction

You have already studied about triangles, and in particular, right triangles, in your earlier classes. Let us take some examples from our surroundings where right triangles can be imagined to be formed. For instance :

1. Suppose the students of a school are visiting Qutub Minar. Now, if a student is looking at the top of the Minar, a right triangle can be imagined to be made, as shown in Fig 8.1. Can the student find out the height of the Minar, without actually measuring it?

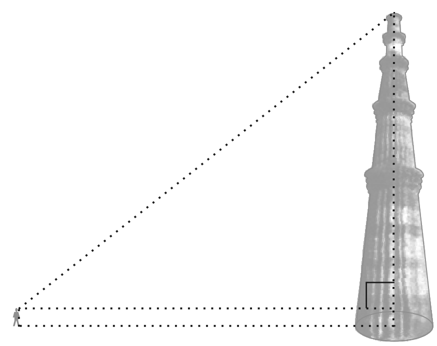


Fig. 8.1

2. Suppose a girl is sitting on the balcony of her house located on the bank of a river. She is looking down at a flower pot placed on a stair of a temple situated nearby on the other bank of the river. A right triangle is imagined to be made in this situation as shown in Fig.8.2. If you know the height at which the person is sitting, can you find the width of the river?.

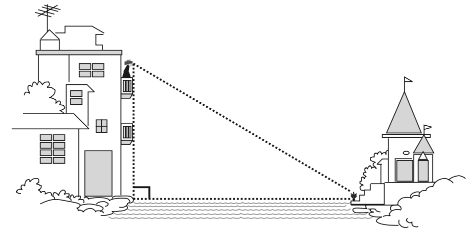


Fig. 8.2

3. Suppose a hot air balloon is flying in the air. A girl happens to spot the balloon in the sky and runs to her mother to tell her about it. Her mother rushes out of the house to look at the balloon.Now when the girl had spotted the balloon intially it was at point A. When both the mother and daughter came out to see it, it had already travelled to another point B. Can you find the altitude of B from the ground?

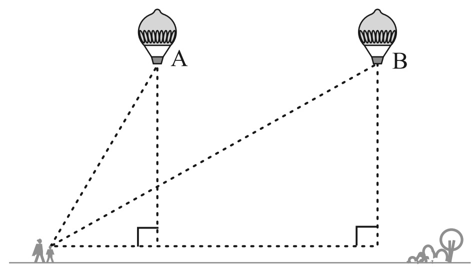


Fig. 8.3

In all the situations given above, the distances or heights can be found by using some mathematical techniques, which come under a branch of mathematics called 'trigonometry'. The word 'trigonometry' is derived from the Greek words 'tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, **trigonometry** is the study of relationships between the sides and angles of a triangle. The earliest known work on trigonometry was recorded in Egypt and Babylon. Early astronomers used it to find out the distances of the stars and planets from the Earth. Even today, most of the technologically advanced methods used in Engineering and Physical Sciences are based on trigonometrical concepts.

In this chapter, we will study some ratios of the sides of a right triangle with respect to its acute angles, called **trigonometric ratios of the angle.** We will restrict our discussion to acute angles only. However, these ratios can be extended to other angles also. We will also define the trigonometric ratios for angles of measure 0° and 90°. We will calculate trigonometric ratios for some specific angles and establish some identities involving these ratios, called **trigonometric identities.**

8.2 Trigonometric Ratios

In Section 8.1, you have seen some right triangles imagined to be formed in different situations.

Let us take a right triangle ABC as shown in Fig. 8.4.

Here, $\angle$ CAB (or, in brief, angle A) is an acute angle. Note the position of the side BC with respect to angle A. It faces $\angle$ A. We call it the *side opposite* to angle A. AC is the *hypotenuse* of the right triangle and the side AB is a part of $\angle$ A. So, we call it the *side adjacent* to angle A.

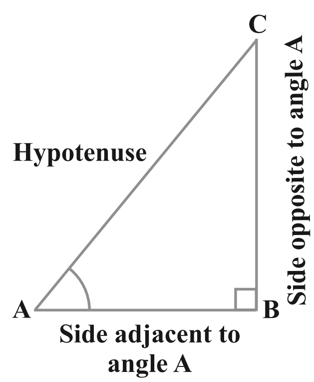


Fig. 8.4

Note that the position of sides change when you consider angle C in place of A (see Fig. 8.5).

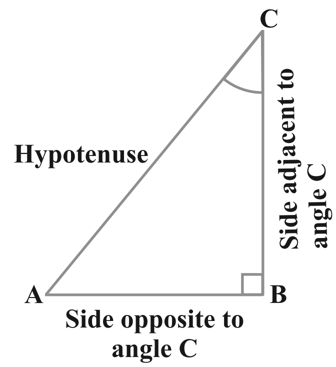


Fig. 8.5

You have studied the concept of 'ratio' in your earlier classes. We now define certain ratios involving the sides of a right triangle, and call them trigonometric ratios.

The trigonometric ratios of the angle A in right triangle ABC (see Fig. 8.4) are defined as follows :

$$\text{sine of } \angle A = \frac{\text{side opposite to angle A}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \frac{\text{side adjacent to angle A}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \frac{\text{side opposite to angle A}}{\text{side adjacent to angle A}} = \frac{BC}{AB}$$

$$\text{cosecant of } \angle A = \frac{1}{\text{sine of } \angle A} = \frac{\text{hypotenuse}}{\text{side opposite to angle A}} = \frac{AC}{BC}$$

$$\text{secant of } \angle A = \frac{1}{\text{cosine of } \angle A} = \frac{\text{hypotenuse}}{\text{side adjacent to angle A}} = \frac{AC}{BC}$$

$$\text{cotangent of } \angle A = \frac{1}{\text{tangent of } \angle A} = \frac{\text{side adjacent to angle A}}{\text{side opposite to angle A}} = \frac{AB}{BC}$$

The ratios defined above are abbreviated as sin A, cos A, tan A, cosec A, sec A and cot A respectively. Note that the ratios **cosec A, sec A and cot A** are respectively, the reciprocals of the ratios sin A, cos A and tan A.

$$\text{Also, observe that } \tan A = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin A}{\cos A} \text{ and } \cot A = \frac{\cos A}{\sin A}.$$

So, the **trigonometric ratios** of an acute angle in a right triangle express the relationship between the angle and the length of its sides.

Why don't you try to define the trigonometric ratios for angle C in the right triangle? (See Fig. 8.5)

The first use of the idea of **'sine'** in the way we use it today was in the work *Aryabhatiyam* by Aryabhatta, in A.D. 500. Aryabhatta used the word ***ardha-jya*** for the half-chord, which was shortened to ***jya*** or ***jiva*** in due course. When the *Aryabhatiyam* was translated into Arabic, the word ***jiva*** was retained as it is. The word ***jiva*** was translated into ***sinus,*** which means curve, when the Arabic version was translated into Latin. Soon the word ***sinus***, also used as ***sine***, became common in mathematical texts throughout Europe. An English Professor of astronomy Edmund Gunter (1581–1626), first used the abbreviated notation **'*sin*'.**

Aryabhatta
A.D. 476 – 550

The origin of the terms **'cosine'** and **'tangent'** was much later. The cosine function arose from the need to compute the sine of the complementary angle. Aryabhatta called it **kotijya**. The name ***cosinus*** originated with Edmund Gunter. In 1674, the English Mathematician Sir Jonas Moore first used the abbreviated notation **'*cos*'**.

Remark : Note that the symbol sin A is used as an abbreviation for 'the sine of the angle A'. sin A is *not* the product of 'sin' and A. 'sin' separated from A has no meaning. Similarly, cos A is *not* the product of 'cos' and A. Similar interpretations follow for other trigonometric ratios also.

Now, if we take a point P on the hypotenuse AC or a point Q on AC extended, of the right triangle ABC and draw PM perpendicular to AB and QN perpendicular to AB extended (see Fig. 8.6), how will the trigonometric ratios of $\angle$ A in Δ PAM differ from those of $\angle$ A in Δ CAB or from those of $\angle$ A in Δ QAN?

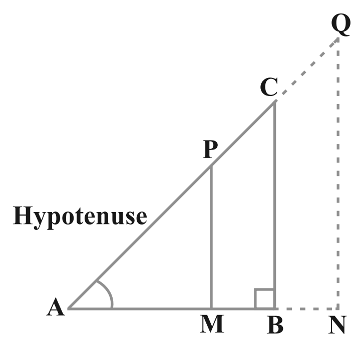


Fig. 8.6

To answer this, first look at these triangles. Is Δ PAM similar to Δ CAB? From Chapter 6, recall the AA similarity criterion. Using the criterion, you will see that the triangles PAM and CAB are similar. Therefore, by the property of similar triangles, the corresponding sides of the triangles are proportional.

So, we have $$\frac{\text{AM}}{\text{AB}} = \frac{\text{AP}}{\text{AC}} = \frac{\text{MP}}{\text{BC}}.$$

From this, we find $\frac{MP}{AP} = \frac{BC}{AC} = \sin A$.

Similarly, $\frac{AM}{AP} = \frac{AB}{AC} = \cos A$, $\frac{MP}{AM} = \frac{BC}{AB} = \tan A$ and so on.

This shows that the trigonometric ratios of angle A in Δ PAM do not differ from those of angle A in Δ CAB.

In the same way, you should check that the value of sin A (and also of other trigonometric ratios) remains the same in Δ QAN also.

From our observations, it is now clear that **the values of the trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle, if the angle remains the same.**

Note : For the sake of convenience, we may write $\sin^2 A$, $\cos^2 A$, etc., in place of $(\sin A)^2$, $(\cos A)^2$, etc., respectively. But cosec A = $(\sin A)^{-1} \neq \sin^{-1} A$ (it is called sine inverse A). $\sin^{-1} A$ has a different meaning, which will be discussed in higher classes. Similar conventions hold for the other trigonometric ratios as well. Sometimes, the Greek letter θ (theta) is also used to denote an angle.

We have defined six trigonometric ratios of an acute angle. If we know any one of the ratios, can we obtain the other ratios? Let us see.

If in a right triangle ABC, $\sin A = \frac{1}{3}$, then this means that $\frac{BC}{AC} = \frac{1}{3}$, i.e., the lengths of the sides BC and AC of the triangle ABC are in the ratio 1 : 3 (see Fig. 8.7). So if BC is equal to k, then AC will be $3k$, where k is any positive number. To determine other trigonometric ratios for the angle A, we need to find the length of the third side AB. Do you remember the Pythagoras theorem? Let us use it to determine the required length AB.

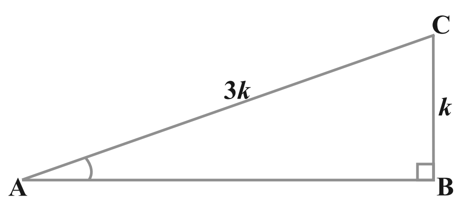


Fig. 8.7

$$AB^2 = AC^2 - BC^2 = (3k)^2 - (k)^2 = 8k^2 = (2\sqrt{2}\,k)^2$$

Therefore, $AB = \pm 2\sqrt{2}\,k$

So, we get $AB = 2\sqrt{2}\,k$ (Why is AB not $-2\sqrt{2}\,k$?)

Now, $\cos A = \frac{AB}{AC} = \frac{2\sqrt{2}\,k}{3k} = \frac{2\sqrt{2}}{3}$

Similarly, you can obtain the other trigonometric ratios of the angle A.

Remark : Since the hypotenuse is the longest side in a right triangle, the value of sin A or cos A is always less than 1 (or, in particular, equal to 1).

Let us consider some examples.

Example 1 : Given tan A = $\frac{4}{3}$, find the other trigonometric ratios of the angle A.

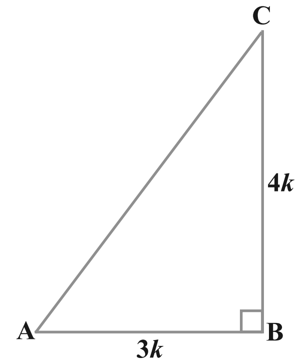


Fig. 8.8

Solution : Let us first draw a right Δ ABC (see Fig 8.8).

Now, we know that $\tan A = \frac{BC}{AB} = \frac{4}{3}$.

Therefore, if BC = $4k$, then AB = $3k$, where k is a positive number.

Now, by using the Pythagoras Theorem, we have

$$AC^2 = AB^2 + BC^2 = (4k)^2 + (3k)^2 = 25k^2$$

So, $\quad AC = 5k$

Now, we can write all the trigonometric ratios using their definitions.

$$\sin A = \frac{BC}{AC} = \frac{4k}{5k} = \frac{4}{5}$$

$$\cos A = \frac{AB}{AC} = \frac{3k}{5k} = \frac{3}{5}$$

Therefore, $\cot A = \frac{1}{\tan A} = \frac{3}{4}$, $\text{cosec } A = \frac{1}{\sin A} = \frac{5}{4}$ and $\sec A = \frac{1}{\cos A} = \frac{5}{3}$.

Example 2 : If ∠ B and ∠ Q are acute angles such that sin B = sin Q, then prove that ∠ B = ∠ Q.

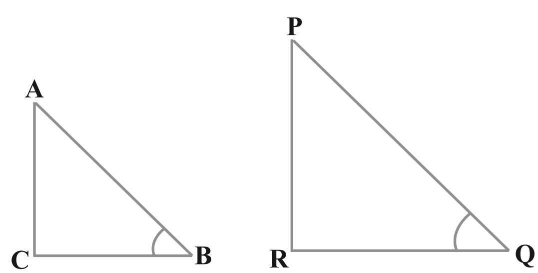


Fig. 8.9

Solution : Let us consider two right triangles ABC and PQR where sin B = sin Q (see Fig. 8.9).

We have $\quad \sin B = \frac{AC}{AB}$

and $\quad \sin Q = \frac{PR}{PQ}$

Then $$\frac{AC}{AB} = \frac{PR}{PQ}$$

Therefore, $$\frac{AC}{PR} = \frac{AB}{PQ} = k, \text{ say} \qquad (1)$$

Now, using Pythagoras theorem,

$$BC = \sqrt{AB^2 - AC^2}$$

and $$QR = \sqrt{PQ^2 - PR^2}$$

So, $$\frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}} = \frac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}} = \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k \qquad (2)$$

From (1) and (2), we have

$$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$$

Then, by using Theorem 6.4, Δ ACB ~ Δ PRQ and therefore, ∠ B = ∠ Q.

Example 3 : Consider Δ ACB, right-angled at C, in which AB = 29 units, BC = 21 units and ∠ ABC = θ (see Fig. 8.10). Determine the values of

(i) $\cos^2 \theta + \sin^2 \theta$,

(ii) $\cos^2 \theta - \sin^2 \theta$.

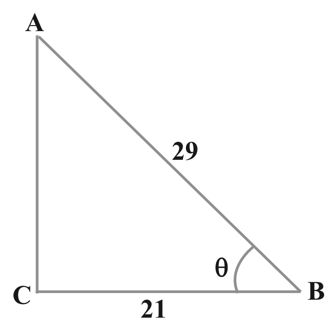


Fig. 8.10

Solution : In Δ ACB, we have

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{(29)^2 - (21)^2}$$

$$= \sqrt{(29 - 21)(29 + 21)} = \sqrt{(8)(50)} = \sqrt{400} = 20 \text{ units}$$

So, $$\sin \theta = \frac{AC}{AB} = \frac{20}{29}, \cos \theta = \frac{BC}{AB} = \frac{21}{29}.$$

Now, (i) $$\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{20^2 + 21^2}{29^2} = \frac{400 + 441}{841} = 1,$$

and (ii) $$\cos^2 \theta - \sin^2 \theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2 = \frac{(21 + 20)(21 - 20)}{29^2} = \frac{41}{841}.$$

Example 4 : In a right triangle ABC, right-angled at B, if tan A = 1, then verify that

2 sin A cos A = 1.

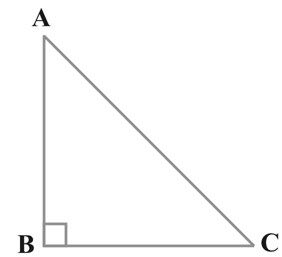


Fig. 8.11

Solution : In Δ ABC, tan A = $\frac{BC}{AB} = 1$ (see Fig 8.11)

i.e., $BC = AB$

Let AB = BC = k, where k is a positive number.

Now, $AC = \sqrt{AB^2 + BC^2}$

$$= \sqrt{(k)^2 + (k)^2} = k\sqrt{2}$$

Therefore, $\sin A = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$ and $\cos A = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$

So, $2 \sin A \cos A = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1,$ which is the required value.

Example 5 : In Δ OPQ, right-angled at P, OP = 7 cm and OQ – PQ = 1 cm (see Fig. 8.12). Determine the values of sin Q and cos Q.

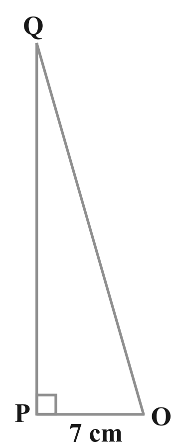


Fig. 8.12

Solution : In Δ OPQ, we have

$$OQ^2 = OP^2 + PQ^2$$

i.e., $(1 + PQ)^2 = OP^2 + PQ^2$ (Why?)

i.e., $1 + PQ^2 + 2PQ = OP^2 + PQ^2$

i.e., $1 + 2PQ = 7^2$ (Why?)

i.e., PQ = 24 cm and OQ = 1 + PQ = 25 cm

So, $\sin Q = \frac{7}{25}$ and $\cos Q = \frac{24}{25}$.

EXERCISE 8.1

1. In Δ ABC, right-angled at B, AB = 24 cm, BC = 7 cm. Determine :
   (i) sin A, cos A
   (ii) sin C, cos C
2. In Fig. 8.13, find tan P – cot R.

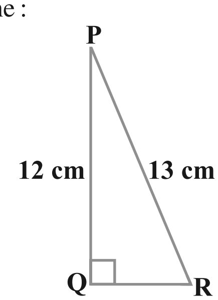


Fig. 8.13

3. If $\sin A = \frac{3}{4}$, calculate cos A and tan A.
4. Given 15 cot A = 8, find sin A and sec A.
5. Given $\sec \theta = \frac{13}{12}$, calculate all other trigonometric ratios.
6. If $\angle$ A and $\angle$ B are acute angles such that cos A = cos B, then show that $\angle$ A = $\angle$ B.
7. If $\cot \theta = \frac{7}{8}$, evaluate : (i) $\frac{(1 + \sin \theta)(1 - \sin \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$, (ii) $\cot^2 \theta$
8. If 3 cot A = 4, check whether $\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos^2 A - \sin^2 A$ or not.
9. In triangle ABC, right-angled at B, if $\tan A = \frac{1}{\sqrt{3}}$, find the value of:
   (i) sin A cos C + cos A sin C
   (ii) cos A cos C – sin A sin C
10. In Δ PQR, right-angled at Q, PR + QR = 25 cm and PQ = 5 cm. Determine the values of sin P, cos P and tan P.
11. State whether the following are true or false. Justify your answer.
    (i) The value of tan A is always less than 1.
    (ii) $\sec A = \frac{12}{5}$ for some value of angle A.
    (iii) cos A is the abbreviation used for the cosecant of angle A.
    (iv) cot A is the product of cot and A.
    (v) $\sin \theta = \frac{4}{3}$ for some angle θ.

8.3 Trigonometric Ratios of Some Specific Angles

From geometry, you are already familiar with the construction of angles of 30°, 45°, 60° and 90°. In this section, we will find the values of the trigonometric ratios for these angles and, of course, for 0°.

Trigonometric Ratios of 45°

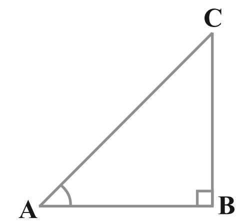


Fig. 8.14

In Δ ABC, right-angled at B, if one angle is 45°, then the other angle is also 45°, i.e., ∠ A = ∠ C = 45° (see Fig. 8.14).

So, BC = AB (Why?)

Now, Suppose BC = AB = a.

Then by Pythagoras Theorem, $AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$,

and, therefore, $AC = a\sqrt{2}$.

Using the definitions of the trigonometric ratios, we have :

$$\sin 45° = \frac{\text{side opposite to angle } 45°}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{\text{side adjacent to angle } 45°}{\text{hypotenuse}} = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{\text{side opposite to angle } 45°}{\text{side adjacent to angle } 45°} = \frac{BC}{AB} = \frac{a}{a} = 1$$

Also, $\text{cosec } 45° = \frac{1}{\sin 45°} = \sqrt{2}$, $\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}$, $\cot 45° = \frac{1}{\tan 45°} = 1$.

Trigonometric Ratios of 30° and 60°

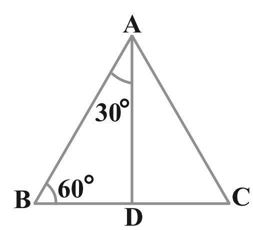


Fig. 8.15

Let us now calculate the trigonometric ratios of 30° and 60°. Consider an equilateral triangle ABC. Since each angle in an equilateral triangle is 60°, therefore, ∠ A = ∠ B = ∠ C = 60°.

Draw the perpendicular AD from A to the side BC (see Fig. 8.15).

Now Δ ABD ≅ Δ ACD (Why?)

Therefore, BD = DC

and ∠ BAD = ∠ CAD (CPCT)

Now observe that:

Δ ABD is a right triangle, right-angled at D with ∠ BAD = 30° and ∠ ABD = 60° (see Fig. 8.15).

As you know, for finding the trigonometric ratios, we need to know the lengths of the sides of the triangle. So, let us suppose that AB = $2a$.

Then, $$\text{BD} = \frac{1}{2}\text{BC} = a$$

and $$\text{AD}^2 = \text{AB}^2 - \text{BD}^2 = (2a)^2 - (a)^2 = 3a^2,$$

Therefore, $$\text{AD} = a\sqrt{3}$$

Now, we have :

$$\sin 30^\circ = \frac{\text{BD}}{\text{AB}} = \frac{a}{2a} = \frac{1}{2},\ \cos 30^\circ = \frac{\text{AD}}{\text{AB}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{\text{BD}}{\text{AD}} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Also, $$\text{cosec } 30^\circ = \frac{1}{\sin 30^\circ} = 2,\quad \sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}.$$

Similarly,

$$\sin 60^\circ = \frac{\text{AD}}{\text{AB}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},\ \cos 60^\circ = \frac{1}{2},\ \tan 60^\circ = \sqrt{3},$$

$$\text{cosec } 60^\circ = \frac{2}{\sqrt{3}},\ \sec 60^\circ = 2 \text{ and } \cot 60^\circ = \frac{1}{\sqrt{3}}.$$

Trigonometric Ratios of 0° and 90°

Let us see what happens to the trigonometric ratios of angle A, if it is made smaller and smaller in the right triangle ABC (see Fig. 8.16), till it becomes zero. As ∠ A gets smaller and smaller, the length of the side BC decreases.The point C gets closer to point B, and finally when ∠ A becomes very close to 0°, AC becomes almost the same as AB (see Fig. 8.17).

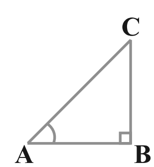


Fig. 8.16

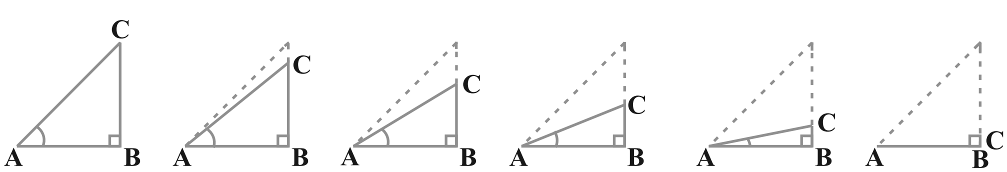


Fig. 8.17

When $\angle$ A is very close to 0°, BC gets very close to 0 and so the value of $\sin A = \frac{BC}{AC}$ is very close to 0. Also, when $\angle$ A is very close to 0°, AC is nearly the same as AB and so the value of $\cos A = \frac{AB}{AC}$ is very close to 1.

This helps us to see how we can define the values of sin A and cos A when A = 0°. We define : **sin 0° = 0 and cos 0° = 1.**

Using these, we have :

$\tan 0° = \frac{\sin 0°}{\cos 0°} = 0$, $\cot 0° = \frac{1}{\tan 0°}$, which is not defined. (Why?)

$\sec 0° = \frac{1}{\cos 0°} = 1$ and $\operatorname{cosec} 0° = \frac{1}{\sin 0°}$, which is again not defined.(Why?)

Now, let us see what happens to the trigonometric ratios of $\angle$ A, when it is made larger and larger in Δ ABC till it becomes 90°. As $\angle$ A gets larger and larger, $\angle$ C gets smaller and smaller. Therefore, as in the case above, the length of the side AB goes on decreasing. The point A gets closer to point B. Finally when $\angle$ A is very close to 90°, $\angle$ C becomes very close to 0° and the side AC almost coincides with side BC (see Fig. 8.18).

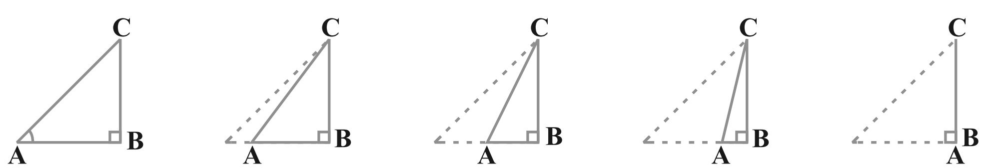


Fig. 8.18

When $\angle$ C is very close to 0°, $\angle$ A is very close to 90°, side AC is nearly the same as side BC, and so sin A is very close to 1. Also when $\angle$ A is very close to 90°, $\angle$ C is very close to 0°, and the side AB is nearly zero, so cos A is very close to 0.

So, we define : **sin 90° = 1 and cos 90° = 0.**

Now, why don't you find the other trigonometric ratios of 90°?

We shall now give the values of all the trigonometric ratios of 0°, 30°, 45°, 60° and 90° in Table 8.1, for ready reference.

Table 8.1

$\angle$ A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Remark : From the table above you can observe that as $\angle$ A increases from 0° to 90°, sin A increases from 0 to 1 and cos A decreases from 1 to 0.

Let us illustrate the use of the values in the table above through some examples.

Example 6 : In Δ ABC, right-angled at B, AB = 5 cm and $\angle$ ACB = 30° (see Fig. 8.19). Determine the lengths of the sides BC and AC.

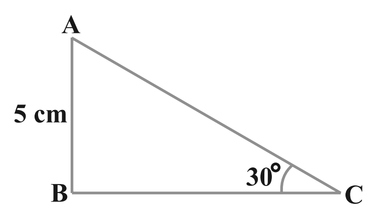


Fig. 8.19

Solution : To find the length of the side BC, we will choose the trigonometric ratio involving BC and the given side AB. Since BC is the side adjacent to angle C and AB is the side opposite to angle C, therefore

$$\frac{AB}{BC} = \tan C$$

i.e.,
$$\frac{5}{BC} = \tan 30° = \frac{1}{\sqrt{3}}$$

which gives
$$BC = 5\sqrt{3} \text{ cm}$$

To find the length of the side AC, we consider

$$\sin 30° = \frac{AB}{AC} \qquad \text{(Why?)}$$

i.e.,
$$\frac{1}{2} = \frac{5}{AC}$$

i.e.,
$$AC = 10 \text{ cm}$$

Note that alternatively we could have used Pythagoras theorem to determine the third side in the example above,

i.e.,
$$AC = \sqrt{AB^2 + BC^2} = \sqrt{5^2 + (5\sqrt{3})^2} \text{ cm} = 10 \text{cm}.$$

Example 7 : In Δ PQR, right-angled at Q (see Fig. 8.20), PQ = 3 cm and PR = 6 cm. Determine ∠ QPR and ∠ PRQ.

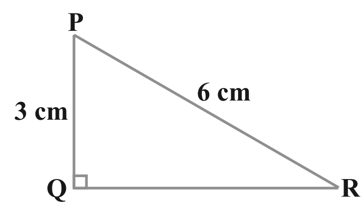


Fig. 8.20

Solution : Given PQ = 3 cm and PR = 6 cm.

Therefore,
$$\frac{PQ}{PR} = \sin R$$

or
$$\sin R = \frac{3}{6} = \frac{1}{2}$$

So,
$$\angle PRQ = 30°$$

and therefore,
$$\angle QPR = 60°. \quad \text{(Why?)}$$

You may note that if one of the sides and any other part (either an acute angle or any side) of a right triangle is known, the remaining sides and angles of the triangle can be determined.

Example 8 : If $\sin (A - B) = \frac{1}{2}$, $\cos (A + B) = \frac{1}{2}$, $0° < A + B \leq 90°$, $A > B$, find A and B.

Solution : Since, $\sin (A - B) = \frac{1}{2}$, therefore, $A - B = 30°$ (Why?) (1)

Also, since $\cos (A + B) = \frac{1}{2}$, therefore, $A + B = 60°$ (Why?) (2)

Solving (1) and (2), we get : $A = 45°$ and $B = 15°$.

EXERCISE 8.2

1. Evaluate the following :

(i) $\sin 60° \cos 30° + \sin 30° \cos 60°$ (ii) $2 \tan^2 45° + \cos^2 30° - \sin^2 60°$

(iii) $\dfrac{\cos 45°}{\sec 30° + \operatorname{cosec} 30°}$ (iv) $\dfrac{\sin 30° + \tan 45° - \operatorname{cosec} 60°}{\sec 30° + \cos 60° + \cot 45°}$

(v) $\dfrac{5 \cos^2 60° + 4 \sec^2 30° - \tan^2 45°}{\sin^2 30° + \cos^2 30°}$

2. Choose the correct option and justify your choice :

(i) $\dfrac{2 \tan 30°}{1 + \tan^2 30°} =$

(A) $\sin 60°$ (B) $\cos 60°$ (C) $\tan 60°$ (D) $\sin 30°$

(ii) $\dfrac{1 - \tan^2 45°}{1 + \tan^2 45°} =$

(A) $\tan 90°$ (B) 1 (C) $\sin 45°$ (D) 0

(iii) $\sin 2A = 2 \sin A$ is true when $A =$

(A) $0°$ (B) $30°$ (C) $45°$ (D) $60°$

(iv) $\dfrac{2 \tan 30°}{1 - \tan^2 30°} =$

(A) $\cos 60°$ (B) $\sin 60°$ (C) $\tan 60°$ (D) $\sin 30°$

3. If $\tan (A + B) = \sqrt{3}$ and $\tan (A - B) = \dfrac{1}{\sqrt{3}}$; $0° < A + B \le 90°$; $A > B$, find A and B.

4. State whether the following are true or false. Justify your answer.

(i) $\sin (A + B) = \sin A + \sin B$.

(ii) The value of $\sin \theta$ increases as θ increases.

(iii) The value of $\cos \theta$ increases as θ increases.

(iv) $\sin \theta = \cos \theta$ for all values of θ.

(v) $\cot A$ is not defined for $A = 0°$.

8.4 Trigonometric Ratios of Complementary Angles

Recall that two angles are said to be complementary if their sum equals 90°. In Δ ABC, right-angled at B, do you see any pair of complementary angles? (See Fig. 8.21)

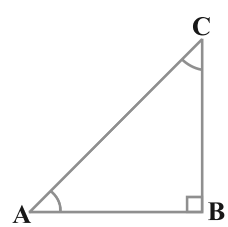


Fig. 8.21

Since $\angle A + \angle C = 90°$, they form such a pair. We have:

$$\left.\begin{array}{lll} \sin A = \dfrac{BC}{AC} & \cos A = \dfrac{AB}{AC} & \tan A = \dfrac{BC}{AB} \\[2ex] \operatorname{cosec} A = \dfrac{AC}{BC} & \sec A = \dfrac{AC}{AB} & \cot A = \dfrac{AB}{BC} \end{array}\right\} \quad (1)$$

Now let us write the trigonometric ratios for $\angle C = 90° - \angle A$.

For convenience, we shall write $90° - A$ instead of $90° - \angle A$.

What would be the side opposite and the side adjacent to the angle $90° - A$?

You will find that AB is the side opposite and BC is the side adjacent to the angle $90° - A$. Therefore,

$$\left.\begin{array}{lll} \sin (90° - A) = \dfrac{AB}{AC}, & \cos (90° - A) = \dfrac{BC}{AC}, & \tan (90° - A) = \dfrac{AB}{BC} \\[2ex] \operatorname{cosec} (90° - A) = \dfrac{AC}{AB}, & \sec (90° - A) = \dfrac{AC}{BC}, & \cot (90° - A) = \dfrac{BC}{AB} \end{array}\right\} \quad (2)$$

Now, compare the ratios in (1) and (2). Observe that :

$$\sin (90° - A) = \frac{AB}{AC} = \cos A \text{ and } \cos (90° - A) = \frac{BC}{AC} = \sin A$$

Also, $$\tan (90° - A) = \frac{AB}{BC} = \cot A, \quad \cot (90° - A) = \frac{BC}{AB} = \tan A$$

$$\sec (90° - A) = \frac{AC}{BC} = \operatorname{cosec} A, \quad \operatorname{cosec} (90° - A) = \frac{AC}{AB} = \sec A$$

So, **sin (90° – A) = cos A,** **cos (90° – A) = sin A,**

tan (90° – A) = cot A, **cot (90° – A) = tan A,**

sec (90° – A) = cosec A, **cosec (90° – A) = sec A,**

for all values of angle A lying between 0° and 90°. Check whether this holds for A = 0° or A = 90°.

Note : tan 0° = 0 = cot 90°, sec 0° = 1 = cosec 90° and sec 90°, cosec 0°, tan 90° and cot 0° are not defined.

Now, let us consider some examples.

Example 9 : Evaluate $\dfrac{\tan 65°}{\cot 25°}$.

Solution : We know : $\cot A = \tan (90° - A)$

So, $\cot 25° = \tan (90° - 25°) = \tan 65°$

i.e., $\dfrac{\tan 65°}{\cot 25°} = \dfrac{\tan 65°}{\tan 65°} = 1$

Example 10 : If sin 3A = cos (A – 26°), where 3A is an acute angle, find the value of A.

Solution : We are given that sin 3A = cos (A – 26°). (1)

Since $\sin 3A = \cos (90° - 3A)$, we can write (1) as

$$\cos (90° - 3A) = \cos (A - 26°)$$

Since 90° – 3A and A – 26° are both acute angles, therefore,

$$90° - 3A = A - 26°$$

which gives $A = 29°$

Example 11 : Express cot 85° + cos 75° in terms of trigonometric ratios of angles between 0° and 45°.

Solution :
$$\cot 85° + \cos 75° = \cot (90° - 5°) + \cos (90° - 15°)$$
$$= \tan 5° + \sin 15°$$

EXERCISE 8.3

1. Evaluate :

   (i) $\dfrac{\sin 18°}{\cos 72°}$ (ii) $\dfrac{\tan 26°}{\cot 64°}$ (iii) $\cos 48° - \sin 42°$ (iv) $\operatorname{cosec} 31° - \sec 59°$

2. Show that :

   (i) $\tan 48° \tan 23° \tan 42° \tan 67° = 1$

   (ii) $\cos 38° \cos 52° - \sin 38° \sin 52° = 0$

3. If tan 2A = cot (A – 18°), where 2A is an acute angle, find the value of A.

4. If tan A = cot B, prove that A + B = 90°.

5. If sec 4A = cosec (A – 20°), where 4A is an acute angle, find the value of A.

6. If A, B and C are interior angles of a triangle ABC, then show that

$$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}.$$

7. Express sin 67° + cos 75° in terms of trigonometric ratios of angles between 0° and 45°.

8.5 Trigonometric Identities

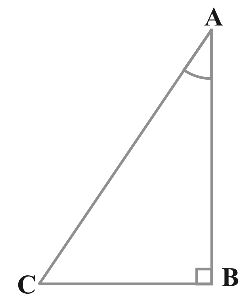


Fig. 8.22

You may recall that an equation is called an identity when it is true for all values of the variables involved. Similarly, an equation involving trigonometric ratios of an angle is called a **trigonometric identity,** if it is true for all values of the angle(s) involved.

In this section, we will prove one trigonometric identity, and use it further to prove other useful trigonometric identities.

In Δ ABC, right-angled at B (see Fig. 8.22), we have:

$$AB^2 + BC^2 = AC^2 \quad (1)$$

Dividing each term of (1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

i.e.,
$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$$

i.e.,
$$(\cos A)^2 + (\sin A)^2 = 1$$

i.e.,
$$\mathbf{\cos^2 A + \sin^2 A = 1} \quad (2)$$

This is true for all A such that 0° ≤ A ≤ 90°. So, this is a trigonometric identity.

Let us now divide (1) by AB^2. We get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

or,
$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e.,
$$\mathbf{1 + \tan^2 A = \sec^2 A} \quad (3)$$

Is this equation true for A = 0°? Yes, it is. What about A = 90°? Well, tan A and sec A are not defined for A = 90°. So, (3) is true for all A such that $0° \le A < 90°$.

Let us see what we get on dividing (1) by BC^2. We get

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

i.e.,
$$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

i.e.,
$$\mathbf{\cot^2 A + 1 = \operatorname{cosec}^2 A} \qquad (4)$$

Note that cosec A and cot A are not defined for A = 0°. Therefore (4) is true for all A such that $0° < A \le 90°$.

Using these identities, we can express each trigonometric ratio in terms of other trigonometric ratios, i.e., if any one of the ratios is known, we can also determine the values of other trigonometric ratios.

Let us see how we can do this using these identities. Suppose we know that $\tan A = \frac{1}{\sqrt{3}}$. Then, $\cot A = \sqrt{3}$.

Since, $\sec^2 A = 1 + \tan^2 A = 1 + \frac{1}{3} = \frac{4}{3}$, $\sec A = \frac{2}{\sqrt{3}}$, and $\cos A = \frac{\sqrt{3}}{2}$.

Again, $\sin A = \sqrt{1 - \cos^2 A} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$. Therefore, cosec A = 2.

Example 12 : Express the ratios cos A, tan A and sec A in terms of sin A.

Solution : Since $\cos^2 A + \sin^2 A = 1$, therefore,

$$\cos^2 A = 1 - \sin^2 A, \text{ i.e., } \cos A = \pm\sqrt{1 - \sin^2 A}$$

This gives
$$\cos A = \sqrt{1 - \sin^2 A} \qquad \text{(Why?)}$$

Hence,
$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}} \text{ and } \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}}$$

Example 13 : Prove that sec A (1 – sin A)(sec A + tan A) = 1.

Solution :

$$\text{LHS} = \sec A\,(1 - \sin A)(\sec A + \tan A) = \left(\frac{1}{\cos A}\right)(1 - \sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1 - \sin A)(1 + \sin A)}{\cos^2 A} = \frac{1 - \sin^2 A}{\cos^2 A}$$

$$= \frac{\cos^2 A}{\cos^2 A} = 1 = \text{RHS}$$

Example 14 : Prove that $\dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$

Solution : $\text{LHS} = \dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\dfrac{\cos A}{\sin A} - \cos A}{\dfrac{\cos A}{\sin A} + \cos A}$

$$= \frac{\cos A\left(\dfrac{1}{\sin A} - 1\right)}{\cos A\left(\dfrac{1}{\sin A} + 1\right)} = \frac{\left(\dfrac{1}{\sin A} - 1\right)}{\left(\dfrac{1}{\sin A} + 1\right)} = \frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1} = \text{RHS}$$

Example 15 : Prove that $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$, using the identity $\sec^2\theta = 1 + \tan^2\theta$.

Solution : Since we will apply the identity involving sec θ and tan θ, let us first convert the LHS (of the identity we need to prove) in terms of sec θ and tan θ by dividing numerator and denominator by cos θ.

$$\text{LHS} = \frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta}$$

$$= \frac{(\tan\theta + \sec\theta) - 1}{(\tan\theta - \sec\theta) + 1} = \frac{\{(\tan\theta + \sec\theta) - 1\}\,(\tan\theta - \sec\theta)}{\{(\tan\theta - \sec\theta) + 1\}\,(\tan\theta - \sec\theta)}$$

$$= \frac{(\tan^2\theta - \sec^2\theta) - (\tan\theta - \sec\theta)}{\{\tan\theta - \sec\theta + 1\}\,(\tan\theta - \sec\theta)}$$

$$= \frac{-1 - \tan\theta + \sec\theta}{(\tan\theta - \sec\theta + 1)\,(\tan\theta - \sec\theta)}$$

$$= \frac{-1}{\tan\theta - \sec\theta} = \frac{1}{\sec\theta - \tan\theta},$$

which is the RHS of the identity, we are required to prove.

EXERCISE 8.4

1. Express the trigonometric ratios sin A, sec A and tan A in terms of cot A.

2. Write all the other trigonometric ratios of $\angle$ A in terms of sec A.

3. Evaluate :

(i) $\dfrac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°}$

(ii) $\sin 25° \cos 65° + \cos 25° \sin 65°$

4. Choose the correct option. Justify your choice.

(i) $9\sec^2 A - 9\tan^2 A =$

(A) 1 (B) 9 (C) 8 (D) 0

(ii) $(1 + \tan\theta + \sec\theta)\,(1 + \cot\theta - \text{cosec}\,\theta) =$

(A) 0 (B) 1 (C) 2 (D) –1

(iii) $(\sec A + \tan A)\,(1 - \sin A) =$

(A) sec A (B) sin A (C) cosec A (D) cos A

(iv) $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} =$

(A) $\sec^2 A$ (B) –1 (C) $\cot^2 A$ (D) $\tan^2 A$

5. Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

(i) $(\text{cosec}\,\theta - \cot\theta)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

(ii) $\dfrac{\cos A}{1 + \sin A} + \dfrac{1 + \sin A}{\cos A} = 2\sec A$

(iii) $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\sec\theta\ \operatorname{cosec}\theta$

[**Hint :** Write the expression in terms of $\sin\theta$ and $\cos\theta$]

(iv) $\dfrac{1+\sec A}{\sec A}=\dfrac{\sin^2 A}{1-\cos A}$ [**Hint :** Simplify LHS and RHS separately]

(v) $\dfrac{\cos A-\sin A+1}{\cos A+\sin A-1}=\operatorname{cosec} A+\cot A$, using the identity $\operatorname{cosec}^2 A = 1+\cot^2 A$.

(vi) $\sqrt{\dfrac{1+\sin A}{1-\sin A}}=\sec A+\tan A$ (vii) $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$

(viii) $(\sin A+\operatorname{cosec} A)^2+(\cos A+\sec A)^2=7+\tan^2 A+\cot^2 A$

(ix) $(\operatorname{cosec} A-\sin A)(\sec A-\cos A)=\dfrac{1}{\tan A+\cot A}$

[**Hint :** Simplify LHS and RHS separately]

(x) $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right)=\left(\dfrac{1-\tan A}{1-\cot A}\right)^2=\tan^2 A$

8.6 Summary

In this chapter, you have studied the following points :

1. In a right triangle ABC, right-angled at B,

$\sin A=\dfrac{\text{side opposite to angle A}}{\text{hypotenuse}}$, $\cos A=\dfrac{\text{side adjacent to angle A}}{\text{hypotenuse}}$

$\tan A=\dfrac{\text{side opposite to angle A}}{\text{side adjacent to angle A}}$.

2. $\operatorname{cosec} A=\dfrac{1}{\sin A}$; $\sec A=\dfrac{1}{\cos A}$; $\tan A=\dfrac{1}{\cot A}$, $\tan A=\dfrac{\sin A}{\cos A}$.

3. If one of the trigonometric ratios of an acute angle is known, the remaining trigonometric ratios of the angle can be easily determined.

4. The values of trigonometric ratios for angles 0°, 30°, 45°, 60° and 90°.

5. The value of sin A or cos A never exceeds 1, whereas the value of sec A or cosec A is always greater than or equal to 1.

6. $\sin(90°-A)=\cos A$, $\cos(90°-A)=\sin A$;

$\tan(90°-A)=\cot A$, $\cot(90°-A)=\tan A$;

$\sec(90°-A)=\operatorname{cosec} A$, $\operatorname{cosec}(90°-A)=\sec A$.

7. $\sin^2 A+\cos^2 A=1$,

$\sec^2 A-\tan^2 A=1$ for $0°\le A<90°$,

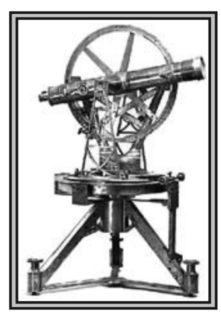
$\operatorname{cosec}^2 A=1+\cot^2 A$ for $0°<A\le 90°$.

SOME APPLICATIONS OF TRIGONOMETRY 9

9.1 Introduction

In the previous chapter, you have studied about trigonometric ratios. In this chapter, you will be studying about some ways in which trigonometry is used in the life around you. Trigonometry is one of the most ancient subjects studied by scholars all over the world. As we have said in Chapter 8, trigonometry was invented because its need arose in astronomy. Since then the astronomers have used it, for instance, to calculate distances from the Earth to the planets and stars. Trigonometry is also used in geography and in navigation. The knowledge of trigonometry is used to construct maps, determine the position of an island in relation to the longitudes and latitudes.

Surveyors have used trigonometry for centuries. One such large surveying project of the nineteenth century was the **'Great Trigonometric Survey'** of British India for which the two largest-ever theodolites were built. During the survey in 1852, the highest mountain in the world was discovered. From a distance of over 160 km, the peak was observed from six different stations. In 1856, this peak was named after Sir George Everest, who had commissioned and first used the giant theodolites (see the figure alongside). The theodolites are now on display in the Museum of the Survey of India in Dehradun.



A Theodolite
(Surveying instrument, which is based on the Principles of trigonometry, is used for measuring angles with a rotating telescope)

In this chapter, we will see how trigonometry is used for finding the heights and distances of various objects, without actually measuring them.

9.2 Heights and Distances

Let us consider Fig. 8.1 of prvious chapter, which is redrawn below in Fig. 9.1.

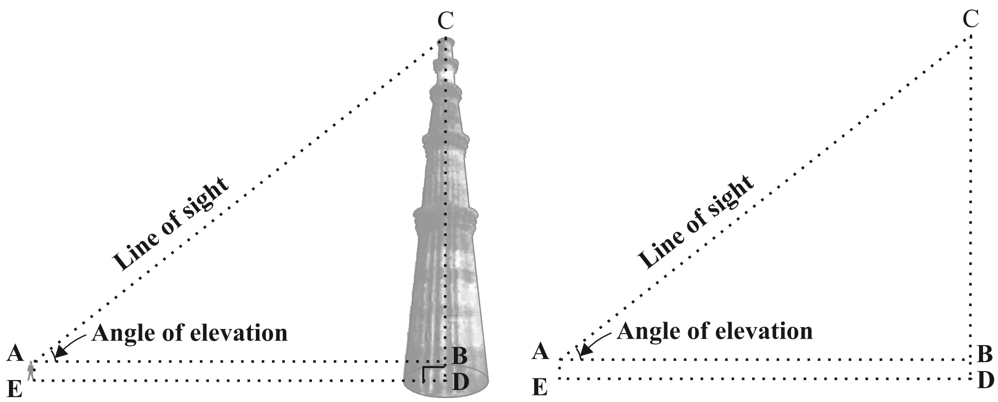


Fig. 9.1

In this figure, the line AC drawn from the eye of the student to the top of the minar is called the *line of sight*. The student is looking at the top of the minar. The angle BAC, so formed by the line of sight with the horizontal, is called the *angle of elevation* of the top of the minar from the eye of the student.

Thus, the **line of sight** is the line drawn from the eye of an observer to the point in the object viewed by the observer. The **angle of elevation** of the point viewed is the angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level, i.e., the case when we raise our head to look at the object (see Fig. 9.2).

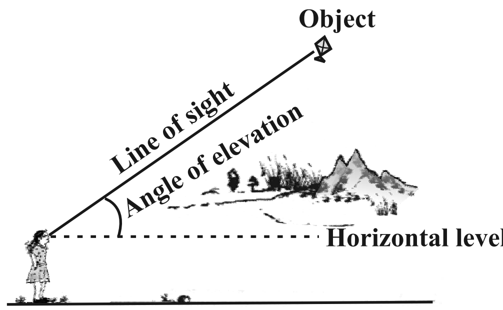


Fig. 9.2

Now, consider the situation given in Fig. 8.2. The girl sitting on the balcony is *looking down* at a flower pot placed on a stair of the temple. In this case, the line of sight is *below* the horizontal level. The angle so formed by the line of sight with the horizontal is called the *angle of depression*.

Thus, the **angle of depression** of a point on the object being viewed is the angle formed by the line of sight with the horizontal when the point is below the horizontal level, i.e., the case when we lower our head to look at the point being viewed (see Fig. 9.3).

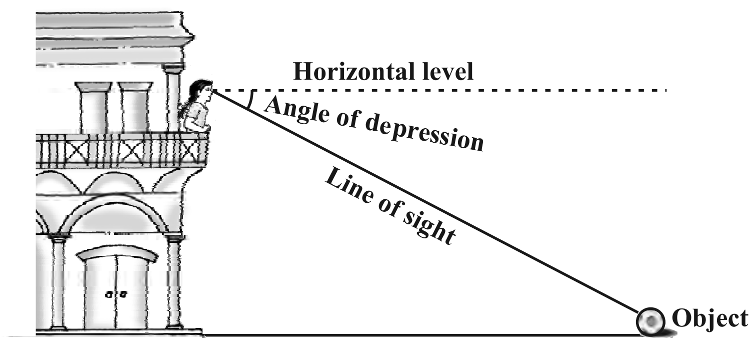


Fig. 9.3

Now, you may identify the lines of sight, and the angles so formed in Fig. 8.3. Are they angles of elevation or angles of depression?

Let us refer to Fig. 9.1 again. If you want to find the height CD of the minar without actually measuring it, what information do you need? You would need to know the following:

(i) The distance DE at which the student is standing from the foot of the minar.

(ii) the angle of elevation, $\angle$ BAC, of the top of the minar.

(iii) the height AE of the student.

Assuming that the above three conditions are known, how can we determine the height of the minar?

In the figure, CD = CB + BD. Here, BD = AE, which is the height of the student.

To find BC, we will use trigonometric ratios of $\angle$ BAC or $\angle$ A.

In Δ ABC, the side BC is the opposite side in relation to the known $\angle$ A. Now, which of the trigonometric ratios can we use? Which one of them has the two values that we have and the one we need to determine? Our search narrows down to using either tan A or cot A, as these ratios involve AB and BC.

Therefore, $\tan A = \frac{BC}{AB}$ or $\cot A = \frac{AB}{BC}$, which on solving would give us BC.

By adding AE to BC, you will get the height of the minar.

Now let us explain the process, we have just discussed, by solving some problems.

Example 1 : A tower stands vertically on the ground. From a point on the ground, which is 15 m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.

Solution : First let us draw a simple diagram to represent the problem (see Fig. 9.4). Here AB represents the tower, CB is the distance of the point from the tower and $\angle$ ACB is the angle of elevation. We need to determine the height of the tower, i.e., AB. Also, ACB is a triangle, right-angled at B.

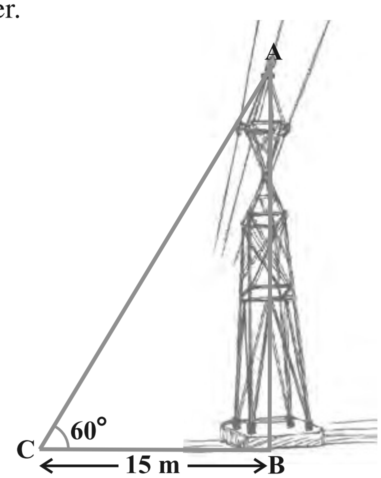


Fig. 9.4

To solve the problem, we choose the trigonometric ratio tan 60° (or cot 60°), as the ratio involves AB and BC.

Now, $\tan 60° = \frac{AB}{BC}$

i.e., $\sqrt{3} = \frac{AB}{15}$

i.e., $AB = 15\sqrt{3}$

Hence, the height of the tower is $15\sqrt{3}$ m.

Example 2 : An electrician has to repair an electric fault on a pole of height 5 m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work (see Fig. 9.5). What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder? (You may take $\sqrt{3}$ = 1.73)

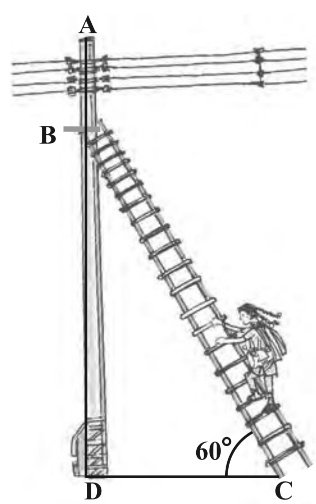


Fig. 9.5

Solution : In Fig. 9.5, the electrician is required to reach the point B on the pole AD.

So, $\quad BD = AD - AB = (5 - 1.3)\text{m} = 3.7 \text{ m}.$

Here, BC represents the ladder. We need to find its length, i.e., the hypotenuse of the right triangle BDC.

Now, can you think which trigonometic ratio should we consider?

It should be sin 60°.

So, $$\frac{BD}{BC} = \sin 60° \text{ or } \frac{3.7}{BC} = \frac{\sqrt{3}}{2}$$

Therefore, $$BC = \frac{3.7 \times 2}{\sqrt{3}} = 4.28 \text{ m (approx.)}$$

i.e., the length of the ladder should be 4.28 m.

Now, $$\frac{DC}{BD} = \cot 60° = \frac{1}{\sqrt{3}}$$

i.e., $$DC = \frac{3.7}{\sqrt{3}} = 2.14 \text{ m (approx.)}$$

Therefore, she should place the foot of the ladder at a distance of 2.14 m from the pole.

Example 3 : An observer 1.5 m tall is 28.5 m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?

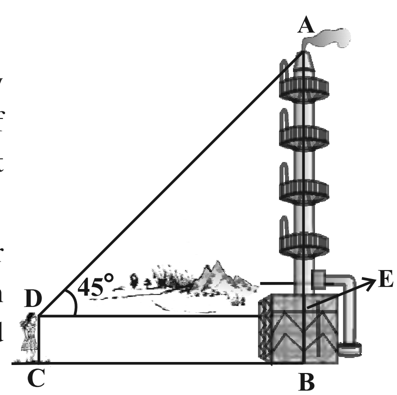


Fig. 9.6

Solution : Here, AB is the chimney, CD the observer and ∠ ADE the angle of elevation (see Fig. 9.6). In this case, ADE is a triangle, right-angled at E and we are required to find the height of the chimney.

We have $\quad AB = AE + BE = AE + 1.5$

and $\quad DE = CB = 28.5 \text{ m}$

To determine AE, we choose a trigonometric ratio, which involves both AE and DE. Let us choose the tangent of the angle of elevation.

Now, $\tan 45° = \frac{AE}{DE}$

i.e., $1 = \frac{AE}{28.5}$

Therefore, AE = 28.5

So the height of the chimney (AB) = (28.5 + 1.5) m = 30 m.

Example 4 : From a point P on the ground the angle of elevation of the top of a 10 m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from P is 45°. Find the length of the flagstaff and the distance of the building from the point P. (You may take $\sqrt{3}$ = 1.732)

Solution : In Fig. 9.7, AB denotes the height of the building, BD the flagstaff and P the given point. Note that there are two right triangles PAB and PAD. We are required to find the length of the flagstaff, i.e., DB and the distance of the building from the point P, i.e., PA.

Since, we know the height of the building AB, we will first consider the right Δ PAB.

We have $\tan 30° = \frac{AB}{AP}$

i.e., $\frac{1}{\sqrt{3}} = \frac{10}{AP}$

Therefore, $AP = 10\sqrt{3}$

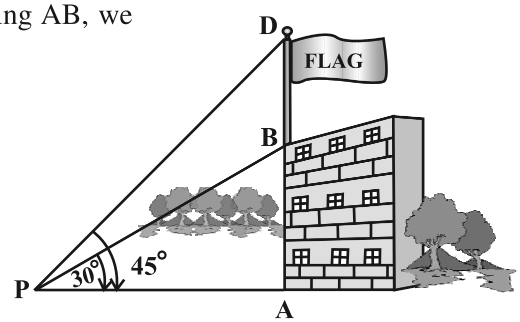


Fig. 9.7

i.e., the distance of the building from P is $10\sqrt{3}$ m = 17.32 m.

Next, let us suppose DB = x m. Then AD = $(10 + x)$ m.

Now, in right Δ PAD, $\tan 45° = \frac{AD}{AP} = \frac{10 + x}{10\sqrt{3}}$

Therefore, $1 = \frac{10 + x}{10\sqrt{3}}$

i.e., $x = 10\left(\sqrt{3}-1\right) = 7.32$

So, the length of the flagstaff is 7.32 m.

Example 5 : The shadow of a tower standing on a level ground is found to be 40 m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

Solution : In Fig. 9.8, AB is the tower and BC is the length of the shadow when the Sun's altitude is 60°, i.e., the angle of elevation of the top of the tower from the tip of the shadow is 60° and DB is the length of the shadow, when the angle of elevation is 30°.

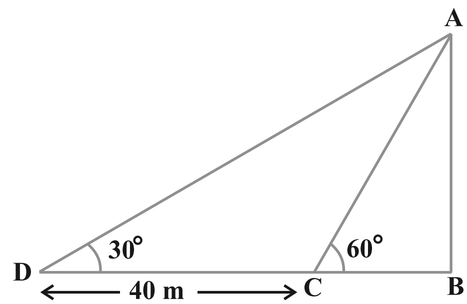


Fig. 9.8

Now, let AB be h m and BC be x m. According to the question, DB is 40 m longer than BC.

So, $\text{DB} = (40 + x)$ m

Now, we have two right triangles ABC and ABD.

In Δ ABC, $\tan 60° = \frac{\text{AB}}{\text{BC}}$

or, $\sqrt{3} = \frac{h}{x}$ (1)

In Δ ABD, $\tan 30° = \frac{\text{AB}}{\text{BD}}$

i.e., $\frac{1}{\sqrt{3}} = \frac{h}{x+40}$ (2)

From (1), we have $h = x\sqrt{3}$

Putting this value in (2), we get $\left(x\sqrt{3}\right)\sqrt{3} = x + 40$, i.e., $3x = x + 40$

i.e., $x = 20$

So, $h = 20\sqrt{3}$ [From (1)]

Therefore, the height of the tower is $20\sqrt{3}$ m.

Example 6 : The angles of depression of the top and the bottom of an 8 m tall building from the top of a multi-storeyed building are 30° and 45°, respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

Solution : In Fig. 9.9, PC denotes the multi-storyed building and AB denotes the 8 m tall building. We are interested to determine the height of the multi-storeyed building, i.e., PC and the distance between the two buildings, i.e., AC.

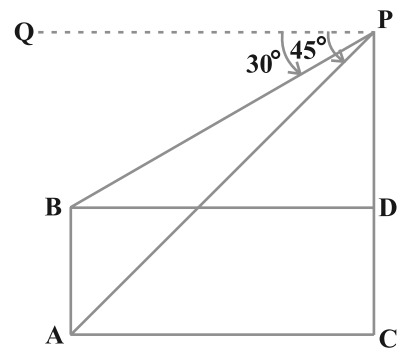


Fig. 9.9

Look at the figure carefully. Observe that PB is a transversal to the parallel lines PQ and BD. Therefore, ∠ QPB and ∠ PBD are alternate angles, and so are equal. So ∠ PBD = 30°. Similarly, ∠ PAC = 45°.

In right Δ PBD, we have

$$\frac{PD}{BD} = \tan 30° = \frac{1}{\sqrt{3}} \quad \text{or} \quad BD = PD\sqrt{3}$$

In right Δ PAC, we have

$$\frac{PC}{AC} = \tan 45° = 1$$

i.e., $\quad PC = AC$

Also, $\quad PC = PD + DC$, therefore, $PD + DC = AC$.

Since, AC = BD and DC = AB = 8 m, we get PD + 8 = BD = $PD\sqrt{3}$ (Why?)

This gives $\quad PD = \dfrac{8}{\sqrt{3}-1} = \dfrac{8\left(\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} = 4\left(\sqrt{3}+1\right)$ m.

So, the height of the multi-storeyed building is $\left\{4\left(\sqrt{3}+1\right)+8\right\}$ m $= 4\left(3+\sqrt{3}\right)$ m and the distance between the two buildings is also $4\left(3+\sqrt{3}\right)$ m.

Example 7 : From a point on a bridge across a river, the angles of depression of the banks on opposite sides of the river are 30° and 45°, respectively. If the bridge is at a height of 3 m from the banks, find the width of the river.

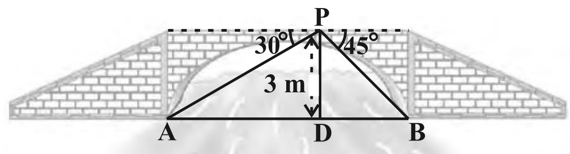


Fig. 9.10

Solution : In Fig 9.10, A and B represent points on the bank on opposite sides of the river, so that AB is the width of the river. P is a point on the bridge at a height of 3 m, i.e., DP = 3 m. We are interested to determine the width of the river, which is the length of the side AB of the Δ APB.

Now, $\quad AB = AD + DB$

In right Δ APD, ∠ A = 30°.

So, $\quad \tan 30° = \frac{PD}{AD}$

i.e., $\quad \frac{1}{\sqrt{3}} = \frac{3}{AD}$ or $AD = 3\sqrt{3}$ m

Also, in right Δ PBD, ∠ B = 45°. So, BD = PD = 3 m.

Now, $\quad AB = BD + AD = 3 + 3\sqrt{3} = 3(1 + \sqrt{3})$ m.

Therefore, the width of the river is $3\left(\sqrt{3}+1\right)$m.

EXERCISE 9.1

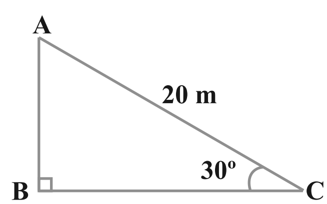


Fig. 9.11

1. A circus artist is climbing a 20 m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30° (see Fig. 9.11).
2. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8 m. Find the height of the tree.
3. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5 m, and

is inclined at an angle of 30° to the ground, whereas for elder children, she wants to have a steep slide at a height of 3m, and inclined at an angle of 60° to the ground. What should be the length of the slide in each case?

4. The angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of the tower, is 30°. Find the height of the tower.

5. A kite is flying at a height of 60 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.

6. A 1.5 m tall boy is standing at some distance from a 30 m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.

7. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are 45° and 60° respectively. Find the height of the tower.

8. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.

9. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.

10. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.

11. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30° (see Fig. 9.12). Find the height of the tower and the width of the canal.

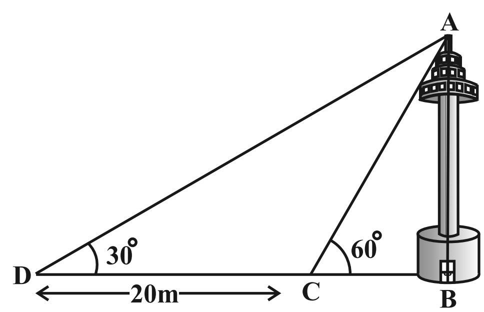


Fig. 9.12

12. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.

13. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.

14. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30° (see Fig. 9.13). Find the distance travelled by the balloon during the interval.

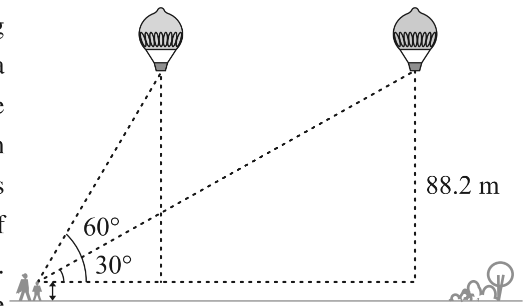


Fig. 9.13

15. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.

16. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.

9.3 Summary

In this chapter, you have studied the following points :

1. (i) The **line of sight** is the line drawn from the eye of an observer to the point in the object viewed by the observer.

(ii) The **angle of elevation** of an object viewed, is the angle formed by the line of sight with the horizontal when it is above the horizontal level, i.e., the case when we raise our head to look at the object.

(iii) The **angle of depression** of an object viewed, is the angle formed by the line of sight with the horizontal when it is below the horizontal level, i.e., the case when we lower our head to look at the object.

2. The height or length of an object or the distance between two distant objects can be determined with the help of trigonometric ratios.

CIRCLES 10

10.1 Introduction

You have studied in Class IX that a circle is a collection of all points in a plane which are at a constant distance (radius) from a fixed point (centre). You have also studied various terms related to a circle like chord, segment, sector, arc etc. Let us now examine the different situations that can arise when a circle and a line are given in a plane.

So, let us consider a circle and a line PQ. There can be three possibilities given in Fig. 10.1 below:

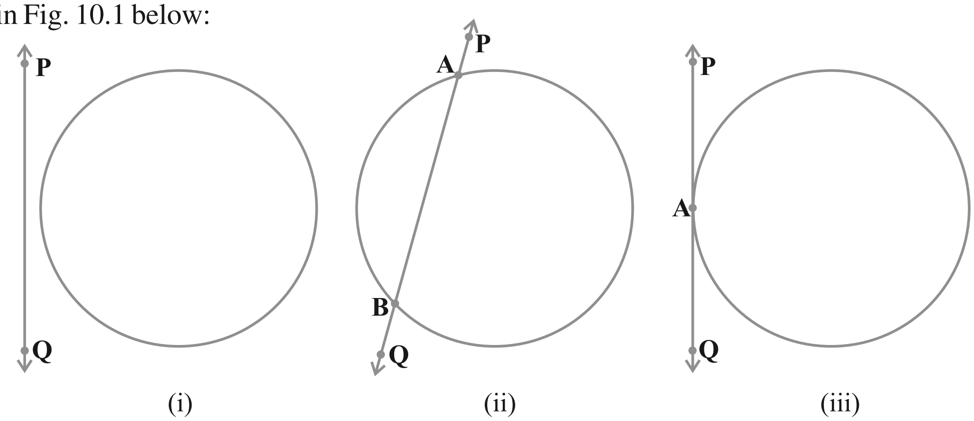


Fig. 10.1

In Fig. 10.1 (i), the line PQ and the circle have no common point. In this case, PQ is called a **non-intersecting** line with respect to the circle. In Fig. 10.1 (ii), there are two common points A and B that the line PQ and the circle have. In this case, we call the line PQ a **secant** of the circle. In Fig. 10.1 (iii), there is only one point A which is common to the line PQ and the circle. In this case, the line is called a **tangent** to the circle.